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Weierstrass semigroups of a pair of points whose first non-gaps are three

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1 Introduction

Let N be the additive semigroup of non-negative integers. Let C be a complete nonsingular curve of genus $g \geq 2$ over an algebraically closed field k of characteristic 0, which is called a *curve* in this talk, and $\mathbf{K}(C)$ the field of rational functions on C.

Definition 1.1. For a point P of C, we set

$$H(P) := \{\alpha \in \mathbf{N} | \text{ there exists } f \in \mathbf{K}(C) \text{ with } (f)_\infty = \alpha P\},$$

which is called the *Weierstrass semigroup of the point* P. An integer n is called the *first non-gap* of P if it is the minimum positive integer in $H(P)$.

Definition 1.2. For distinct points P and Q of C, we set

$$H(P,Q) := \{(\alpha,\beta) \in \mathbf{N} \times \mathbf{N} | \text{ there exists } f \in \mathbf{K}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\},$$

which is called the *Weierstrass semigroup of the pair* (P,Q) *of points.*

Fact 1.3. (Kim [2]) If C is a hyperelliptic curve, i.e., a double covering of the projective line, then the semigroup $H(P,Q)$ is determined explicitly.

Fact 1.4. (Kim-Komeda [4]) If C is of genus 3, then the semigroup $H(P,Q)$ is determined explicitly.

Aim 1.5. Let P and Q be distinct points of C of genus $g \geq 4$. If the first non-gaps of P and Q are three, we determine the semigroup $H(P,Q)$ explicitly. Moreover, we give examples of two pointed curves (C,P,Q) whose semigroups are the given

2 Possible Weierstrass semigroups of genus ≥ 5

First, let us review some Kim's results. Let C be a curve of genus g and P its point.

Definition 2.1. We set

$$G(P) := \mathbf{N} \backslash H(P) = \{l_1 < l_2 < \ldots < l_g\}.$$

The integer l_g is called the *last gap* at P.

Let Q be another point of C which is distinct from P. For each $l \in G(P)$, the integer $\min\{\beta | (l, \beta) \in H(P, Q)\}$ must be equal to some element in $G(Q)$, say $\sigma(l)$, and this correspondence σ gives a bijective map between the sets $G(P)$ and $G(Q)$.

Definition 2.2. We set

$$\Gamma(P, Q) := \{(l, \sigma(l)) | l \in G(P)\}.$$

Fact 2.3. (Kim [2]) The semigroup $H(P, Q)$ is completely determined by the bijective correspondence σ, i.e.,

$$G(P, Q) = \bigcup_{l \in G(P)} (\{(l, \beta) | \beta = 0, 1, \ldots, \sigma(l) - 1\}$$

$$\cup \{(\alpha, \sigma(l)) | \alpha = 0, 1, \ldots, l - 1\}),$$

where we set

$$G(P, Q) = \mathbf{N} \times \mathbf{N} \backslash H(P, Q).$$

Thus, it suffices to determine the graph $\Gamma(P, Q)$ of σ for describing the semigroup $H(P, Q)$.

We consider the case where the first non-gaps of distinct points P and Q are three.

Theorem 2.4. *$\sigma(l_g) = 1$ or $\sigma(l_g) = 2$, i.e., $(l_g, 1) \in \Gamma(P, Q)$ or $(l_g, 2) \in \Gamma(P, Q)$.*

Proof. Assume that $(l_g, \beta) \in \Gamma(P, Q)$ with $\beta \geq 4$. Then

$$(\alpha_1, 1) \in \Gamma(P, Q),\ 1 \leq \alpha_1 < l_g \text{ and } (\alpha_2, 2) \in \Gamma(P, Q),\ 1 \leq \alpha_2 < l_g.$$

Now

$$\beta \equiv i \bmod 3 \text{ for some } i = 1, 2 \text{ and } (\alpha_i, \beta) = (\alpha_i, i) + (0, 3k) \text{ for some } k.$$

Since $(\alpha_i, i) \in \Gamma(P, Q)$ and $(0, 3k) \in H(P, Q)$, we have $(\alpha_i, \beta) \in H(P, Q)$. But $(l_g, \beta) \in \Gamma(P, Q)$ and $\alpha_i < l_g$. This contradicts Fact 2.3. Q.E.D.

From now on we assume that $g \geq 5$. By the theory of curves we can find some integer n with

$$\frac{g-1}{3} \leq n \leq \frac{g}{2}$$

such that

$$S(H(P)) = \{3, 3n+2, 3(g-n)+1\}$$

or

$$S(H(P)) = \{3, 3n+1, 3(g-n)+2\},$$

where $S(H(P)) = \{3, s_1, s_2\}$, which is called the *standard basis* for $H(P)$, with

$$s_i = \mathrm{Min}\{s \in H(P) | s \equiv i \bmod 3\} \text{ for } i = 1, 2.$$

Moreover, since there exists $f \in \mathrm{K}(C)$ such that $(f) = 3P - 3Q$, the standard basis $S(H(Q))$ must be one of the above two.

Definition 2.5. The point P is said to be *of the n-th kind* if

$$S(H(P)) = \{3, 3n+2, 3(g-n)+1\}$$

or

$$S(H(P)) = \{3, 3n+1, 3(g-n)+2\},$$

Definition 2.6. The point P of the n-th kind is said to be *of type I* (resp. *II*) if

$$n \neq \frac{g}{2} \text{ and } 3n+2 \in S(H(P)) \text{ (resp. } 3n+1 \in S(H(P))).$$

We note that If $n = \frac{g}{2}$, then $S(H(P)) = \{3, 3n+1, 3n+2\}$.

Using the types of the points P and Q we can determine whether

$$(l_g, 1) \in \Gamma(P,Q) \text{ or } (l_g, 2) \in \Gamma(P,Q).$$

Theorem 2.7. *Let P and Q be two distinct points of the n-th kind and $n \neq \frac{g}{2}$.*
i) *If P and Q are of type II, then*

$$(l_g, 2) \in \Gamma(P,Q) \text{ and } (3n-2, 1) \in \Gamma(P,Q).$$

ii) *If P (resp. Q) is of type I (resp. II), then*

$$(l_g, 1) \in \Gamma(P,Q) \text{ and } (3n-2, 2) \in \Gamma(P,Q).$$

iii) *If P and Q are of type I, then*

$$(l_g, 1) \in \Gamma(P,Q) \text{ and } (3n-1, 2) \in \Gamma(P,Q).$$

Moreover, if $(\alpha_1, 1)$ *and* $(\alpha_2, 2)$ *belong to* $\Gamma(P,Q)$*, then*

$$\Gamma(P,Q) = \left\{(\alpha_1 - 3k, 1+3k) | 0 \le k < \frac{\alpha_1}{3}\right\} \cup \left\{(\alpha_2 - 3k, 2+3k) | 0 \le k < \frac{\alpha_2}{3}\right\}.$$

In some case with $n = \frac{g}{3}$ we have no candidate of the semigroup $H(P,Q)$.

Proposition 2.8. *Let* P *and* Q *be points of the* $\frac{g}{3}$*-th kind. If they are of type II, then* $P = Q$*. In this case,* $H(P)$ *is generated by* 3 *and* $g+1$ *with* $g \equiv 0 \bmod 3$.

There are two possibilities in the case $n = \frac{g}{2}$.

Proposition 2.9. *Let* P *and* Q *be distinct points of the* $\frac{g}{2}$*-th kind, i.e.,*

$$S(H(P)) = S(H(Q)) = \{3, 3n+1, 3n+2\}.$$

Then

$$(3n-1, 1) \in \Gamma(P,Q) \text{ or } (3n-1, 2) \in \Gamma(P,Q).$$

If $(3n-1,1) \in \Gamma(P,Q)$ *(resp.* $(3n-1,2) \in \Gamma(P,Q)$*), then*

$$\Gamma(P,Q) = \{(\alpha, 3n-\alpha) | \alpha \in G(P)\}$$

$$(\text{resp. } \{(3k-2, (3n-1)-(3k-2)) | k = 1, \ldots, n\}$$
$$\cup \{(3k-1, (3n+1)-(3k-1)) | k = 1, \ldots, n\}).$$

3 The existence of two pointed curves

In the previous section we determined the possible Weierstrass semigroups H of a pair of points on a curve of genus≥ 5 whose first non-gaps are three. In this section for each such a semigroup H we give two pointed curves (C, P, Q) such that $H(P,Q) = H$.

Let C be the curve whose function field $\mathbf{K}(C) = k(x,y)$ is defined by the equation

$$y^3 = (x-c_1)\cdots(x-c_{i_1})(x-c_{i_1+1})^2\cdots(x-c_{i_1+i_2})^2,$$

where $c_1, \ldots, c_{i_1+i_2}$ are distinct elements of k and $i_1 + 2i_2$ is not divisible by 3. We note that the genus g of the curve C is $i_1 + i_2 - 1$ by Riemann-Hurwitz formula.

Let $\pi : C \longrightarrow \mathbf{P}^1$ be the morphism corresponding to the inclusion $k(x) \subset \mathbf{K}(C)$, i.e., $\pi(P) = (1 : x(P))$, where $\mathbf{P}^1$ denotes the projective line. We set

$$\{P_\infty\} = \pi^{-1}(0:1) \text{ and } \{P_s\} = \pi^{-1}(1:c_s) \text{ for } s = 1, \ldots, i_1+i_2.$$

Then we have $S(H(P_\infty)) = \{3, i_1 + 2i_2, 2i_1 + i_2\}$ (For example, see Kim-Komeda [3]). If $i_1 + 2i_2 \equiv 1 \bmod 3$,

$$S(H(P_s)) = \begin{cases} \{3, i_1 + 2i_2 + 1, 2i_1 + i_2 - 1\} & \text{if } 1 \leq s \leq i_1 \\ S(H(P_\infty)) = \{3, i_1 + 2i_2, 2i_1 + i_2\} & \text{if } i_1 + 1 \leq s \leq i_1 + i_2 \end{cases}$$

If $i_1 + 2i_2 \equiv 2 \bmod 3$,

$$S(H(P_s)) = \begin{cases} S(H(P_\infty)) = \{3, i_1 + 2i_2, 2i_1 + i_2\} & \text{if } 1 \leq s \leq i_1 \\ \{3, i_1 + 2i_2 - 1, 2i_1 + i_2 + 1\} & \text{if } i_1 + 1 \leq s \leq i_1 + i_2 \end{cases}$$

From now on we assume that $g \geq 5$. By the above formula we get the following examples :

Example 3.1. i) Let $\frac{g}{3} < n < \frac{g}{2}$. If $i_1 = 2g - 3n + 1 > n + 1$ and $i_2 = 3n - g > 0$, then the points P_∞ and P_s $(i_1 + 1 \leq s \leq i_1 + i_2)$ are of the n-th kind of type II.
ii) Let $\frac{g}{3} \leq n \leq \frac{g-1}{2}$. If $i_1 = 2g - 3n + 1 > n + 3$ and $i_2 = 3n - g \geq 0$, then the points P_∞ is of the n-th kind of type II and the points P_s $(1 \leq s \leq i_1)$ are of the n-th kind of type I.
~~iii) Let $\frac{g-1}{3} \leq n \leq \frac{g-1}{2}$. If $i_1 = 2g - 3n \geq n + 2$ and $i_2 = 3n - g + 1 \geq 0$, then~~ the points P_∞ and P_s $(1 \leq s \leq i_1)$ are of the n-th kind of type I.

In the case of Proposition 2.9 we get the following examples:

Example 3.2. Let $n \geq 3$. If $i_1 = n+1$ and $i_2 = n$, then $g = 2n$ and the points P_∞ and P_s are of the $\frac{g}{2}$-th kind. Then $S(H(P_\infty)) = S(H(P_s)) = \{3, 3n + 1, 3n + 2\}$. Moreover,

$$\Gamma(P_\infty, P_s) \ni (3n - 1, 1) \text{ for } 1 \leq s \leq i_1$$

and

$$\Gamma(P_\infty, P_s) \ni (3n - 1, 2) \text{ for } i_1 + 1 \leq s \leq i_1 + i_2.$$

4 Weierstrass semigroups of genus 4

In this section we treat the curves C of genus 4 with point P whose first non-gap is 3. Then we have $S(H(P)) = \{3, 5, 10\}$ or $\{3, 7, 8\}$.

Remark 4.1. Let $S(H(P)) = \{3, 5, 10\}$. If Q is another point of C whose first non-gap is three, then $S(H(Q)) = \{3, 5, 10\}$ and there exists $f \in \mathbf{K}(C)$ such that $(f) = 3P - 3Q$.

Proposition 4.2. *Let P and Q be two distinct points such that $S(H(P)) = S(H(Q)) = \{3, 5, 10\}$. Then $\Gamma(P, Q) = \{(1, 7), (2, 2), (4, 4), (7, 1)\}$. For example, such pointed curves are given by*

$$y^3 = (x - c_1) \cdots (x - c_5),\ P = P_\infty \text{ and } Q = P_s \text{ for } s = 1, \ldots, 5$$

where we use the notations in Section 3.

Proposition 4.3. *Let $S(H(P)) = \{3, 7, 8\}$ and Q another point of C whose first non-gap is three. Suppose that there exists $f \in \mathrm{K}(C)$ such that $(f) = 3P - 3Q$.*
i) $(5, 1) \in \Gamma(P, Q)$ *or* $(5, 2) \in \Gamma(P, Q)$.
ii) *If* $(5, 1) \in \Gamma(P, Q)$, *then* $\Gamma(P, Q) = \{(5, 1), (4, 2), (2, 4), (1, 5)\}$. *For example, such pointed curves are given by*

$$y^3 = (x - c_1)(x - c_2)(x - c_3)(x - c_4)^2(x - c_5)^2,\ P = P_\infty \text{ and } Q = P_s \text{ for } s = 1, 2, 3.$$

iii) *If* $(5, 2) \in \Gamma(P, Q)$, *then* $\Gamma(P, Q) = \{(5, 2), (4, 1), (1, 4), (2, 5)\}$. *Such pointed curves are given by the same equations as above, $P = P_\infty$ and $Q = P_s$ for $s = 4, 5$.*

Proposition 4.4. *Let $S(H(P)) = \{3, 7, 8\}$ and Q another point of C whose first non-gap is three. Suppose that there is no $f \in \mathrm{K}(C)$ such that $(f) = 3P - 3Q$. Then $(5, 1) \in \Gamma(P, Q)$ and $\Gamma(P, Q) = \{(5, 1), (4, 4), (2, 2), (1, 5)\}$. Such curves C are also given by the equations*

$$y^3 = (x - c_1)(x - c_2)(x - c_3)(x - c_4)^2(x - c_5)^2.$$

Using the result of Kato [1] *we get our desired points P and Q.*

References

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